Dan Shanks’ CUFFQI Algorithm Resurrected

Renate Scheidler
rscheidl@ucalgary.ca

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What is CUFFQI?

Short for Cubic Fields From Quadratic Infrastructure

Invented by Dan Shanks (1987)
Editor for Math. Comp. 1959-1996
Made practical and implemented by Gilbert Fung (1990)
Unpublished (to appear as Chapter 4 in Cubic Fields With Geometry by S. Hambleton & H. C. Williams, Springer Monograph 2018/19)
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A cubic field of discriminant $D$ has a generating polynomials of the form

$$f(x) = x^3 - 3N(\lambda)^{1/3}x + Tr(\lambda)$$

- $\lambda$ is an algebraic integer in $\mathbb{Q}(\sqrt{-3D})$
- Norm and trace are taken in $\mathbb{Q}(\sqrt{-3D})/\mathbb{Q}$
- $N(\lambda) \in \mathbb{Z}^3$

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- Norm and trace are taken in $\mathbb{Q}(\sqrt{-3D})/\mathbb{Q}$.
- $N(\lambda) \in \mathbb{Z}^3$ (Berwick 1925)

Roots of $f(x)$ (Cardano 1545):

$$\zeta^i \lambda^{1/3} + \zeta^{-i} \lambda^{1/3} \quad (i = 0, 1, 2)$$

where $\zeta$ is a primitive cube root of unity.
Example: $D = 44806173$

Naively (take $\lambda$ to be the fundamental unit of $\mathbb{Q}(\sqrt{-3 \cdot 44806173})$):

$$f(x) = x^3 - 3x + 9631353811877867340405658366$$
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\[
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\]

Using CUFFQI (all 13 cubic fields with \( D = 44806173 \)):

\[
\begin{align*}
f_1(x) &= x^3 - 61x^2 + 697x - 330 \\
f_2(x) &= x^3 - 279x^2 + 441x - 170 \\
f_3(x) &= x^3 - 63x^2 + 423x - 8 \\
f_4(x) &= x^3 - 69x^2 + 435x - 216 \\
f_5(x) &= x^3 - 63x^2 + 603x - 494 \\
f_6(x) &= x^3 - 83x^2 + 297x - 54 \\
f_7(x) &= x^3 - 63x^2 + 837x - 494 \\
f_8(x) &= x^3 - 257x^2 + 477x - 216 \\
f_9(x) &= x^3 - 87x^2 + 273x - 36 \\
f_{10}(x) &= x^3 - 62x^2 + 546x - 261 \\
f_{11}(x) &= x^3 - 60x^2 + 660x - 97 \\
f_{12}(x) &= x^3 - 165x^2 + 273x - 90 \\
f_{13}(x) &= x^3 - 127x^2 + 185x - 62
\end{align*}
\]
Problem with Berwick construction: polynomial coefficients can be HUGE!

(E.g. $Tr(\varepsilon) \approx \varepsilon \approx \exp(\sqrt{|D|})$ for the fundamental unit $\varepsilon \in \mathbb{Q}(\sqrt{-3D})$)

CUFFQI to the rescue!
Problem with Berwick construction: polynomial coefficients can be HUGE! (E.g. $Tr(\varepsilon) \approx \varepsilon \approx \exp(\sqrt{|D|})$ for the fundamental unit $\varepsilon \in \mathbb{Q}(\sqrt{-3D})$)

**CUFFQI to the rescue!**

**Goal:** for a given fundamental discriminant $D$, produce all the cubic fields of discriminant $D$ à la Berwick via generating polynomials with small coefficients.
The Berwick Map

There is a map from the set of unordered triples of conjugate cubic fields

\[ \{ K, K', K'' \} \quad \text{disc}(K) = D \]

to the set of unordered pairs of 3-torsion ideal classes

\[ \{ [a], [\bar{a}] \} \]

in \( \mathcal{O}_{\mathbb{Q}(\sqrt{-3D})} \) via

\[ x^3 - 3N(\lambda)^{1/3}x + Tr(\lambda) \mapsto \{ [a], [\bar{a}] \} \quad \text{where } a^3 = (\lambda) \]
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For \( D > 0 \):
- bijection onto non-principal ideal classes
- nothing maps to the principal class

For \( D < 0 \):
- 3-to-1 onto non-principal ideal classes
- 1-to-1 onto to the principal class
Some Counting

Put

\[ r = 3\text{-rank}(\text{Cl}(\mathbb{Q}(\sqrt{D}))) \]
\[ s = 3\text{-rank}(\text{Cl}(\mathbb{Q}(\sqrt{-3D}))) \]
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Number of cubic fields of discriminant \( D \) (Hasse 1929):

\[ \frac{3^r - 1}{2} \]
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Number of cubic fields produced by the Berwick map:

For \( D > 0 \):

\[ \frac{3s - 1}{2} \]

For \( D < 0 \):

\[ 3 \cdot \frac{3s - 1}{2} + 1 = \frac{3^{s+1} - 1}{2} \]
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Number of cubic fields produced by the Berwick map:

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\frac{3^s - 1}{2}
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For \(D < 0\):

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3 \cdot \frac{3^s - 1}{2} + 1 = \frac{3^{s+1} - 1}{2}
\]

Connection between \(r\) and \(s\) (Scholz 1932):

\[
|r - s| \leq 1
\]

If \(r \neq s\), then the imaginary quadratic field has the bigger 3-rank
More Counting

Case $D > 0$:

$$r = s: \quad \frac{3^s - 1}{2} = \frac{3^r - 1}{2}$$

$$r = s - 1: \quad \frac{3^s - 1}{2} = \frac{3^r - 1}{2} + 3^r$$

So what are these extra $3r$ cubic fields? Answer: they are the complete collection of cubic fields of discriminant $9D$ if $3 | D$, $81D$ if $3 \nmid D$. In the other cases there are no fields of these discriminants.
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Case $D < 0$:

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So what are these extra $3^r$ cubic fields?

Answer: they are the complete collection of cubic fields of discriminant

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\begin{align*}
9D & \text{ if } 3 \mid D, \\
81D & \text{ if } 3 \nmid D
\end{align*}
\]

In the ☃️ cases there are no fields of these discriminants
Berwick Construction Algorithm

**Input:** $D$ and a basis of $\text{Cl}(\mathbb{Q}(\sqrt{-3D})[3]$

(For $D < 0$, also the regulator $R$ of $\mathbb{Q}(\sqrt{-3D})$)

**Output:** generating polynomials of all cubic fields of discriminant $D$

**Algorithm:**

For each basis class $C$ of $\text{Cl}(\mathbb{Q}(\sqrt{-3D})[3]$, collect generators $\lambda$ of one ideal in $C$ whose cube has a small generator when $D > 0$

three ideals in $C$ whose cube has a small generator when $D < 0$

Collect a small element $\lambda (\notin \mathbb{Z})$ in some principal ideal when $D < 0$

For each $\lambda$ collected

compute $f(x) = x^3 - 3N(\lambda)^{1/3}x + Tr(\lambda)$

if $\text{disc}(f) = D$, output $f(x)$
Reduced Ideals

An ideal \( a \) in \( \mathcal{O}_{\mathbb{Q}(\sqrt{-3D})} \) is **reduced** if no non-zero element \( \alpha \in a \) satisfies

\[
|\alpha| < N(a) \quad \text{and} \quad |\overline{\alpha}| < N(a)
\]

Hence, to get \( \lambda \) of small norm, use reduced ideals (exist in every ideal class).
Reduced Ideals

An ideal $a$ in $\mathcal{O}_{\mathbb{Q}(\sqrt{-3D})}$ is **reduced** if no non-zero element $\alpha \in a$ satisfies

$$|\alpha| < N(a) \quad \text{and} \quad |\bar{\alpha}| < N(a)$$

If $a$ is reduced, then

$$N(a) < \begin{cases} \sqrt{|D'|}/3 & \text{when } D' < 0 \\ \sqrt{D'} & \text{when } D' > 0 \end{cases}$$

where $D' = -D/3$ when $3 \mid D$ and $D' = -3D$ when $3 \nmid D$. 
Reduced Ideals

An ideal \( \alpha \) in \( \mathcal{O}_{\mathbb{Q}(\sqrt{-3D})} \) is **reduced** if no non-zero element \( \alpha \in \alpha \) satisfies

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\end{cases}
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where \( D' = -D/3 \) when \( 3 \mid D \) and \( D' = -3D \) when \( 3 \nmid D \).

If \( \alpha \) is reduced and \( \alpha^3 = (\lambda) \), then

\[
N(\lambda) < \begin{cases} 
(|D'|/3)^{3/2} & \text{when } D' < 0 \\
(D')^{3/2} & \text{when } D' > 0 
\end{cases}
\]

Hence, to get \( \lambda \) of small norm, use reduced ideals (exist in every ideal class)
Generators $\lambda$ of Small Trace, $D' < 0$

Here, the reduced ideal $\alpha$ is unique.
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Write $\lambda = \frac{A + B\sqrt{D'}}{2}$ ($A, B \in \mathbb{Z}$). Then

$$4N(\lambda) = A^2 - B^2D' = A^2 + B^2|D'|$$
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$N(\lambda) < (|D'|/3)^{3/2}$ implies

$$|\text{Tr}(\lambda)| = |A| < \frac{1}{2} \left(\frac{|D'|}{3}\right)^{3/4}$$
Generators $\lambda$ of Small Trace, $D' < 0$

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Happily, the reduced ideal also yields a small trace!
For any ideal class $C$, the **infrastructure** of the $C$ is the collection of all reduced ideals in $C$ (Shanks 1972)
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- Infrastructures are finite.
- Can move from one infrastructure ideal \( \alpha \) to its *neighbour* \( \rho(\alpha) \) via one step in a simple continued fraction expansion
- Infrastructure ideals are discretely spaced on a circle of circumference \( R \), the regulator of \( \mathbb{Q}(\sqrt{D'}) \)
- For any point \( P \) on the circle, there is a unique reduced ideal *closest* to \( P \) (efficiently computable)
Infrastructures, $D' > 0$

Infrastructure of $\mathcal{C} = [\mathcal{t}]$  

$\alpha$ is closest to $P$
\( \lambda \in \mathcal{O}_{\mathbb{Q}(\sqrt{D^\prime})} \) is small if

\[
1 < \lambda < (D^\prime)^{3/2}, \quad |N(\lambda)| < (D^\prime)^{3/2}
\]
Suitable Reduced Ideals, $D' < 0$

$\lambda \in \mathcal{O}_{\mathbb{Q}(\sqrt{D'})}$ is small if

$$1 < \lambda < (D')^{3/2}, \quad |N(\lambda)| < (D')^{3/2}$$

The following reduced ideals have cubes with small generators (Shanks):

- For the principal ideal class, the reduced ideal closest to

  $$R \frac{3}{3} + \log(D') \frac{4}{4}$$
Suitable Reduced Ideals, $D' < 0$

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$$1 < \lambda < (D')^{3/2}, \quad |N(\lambda)| < (D')^{3/2}$$

The following reduced ideals have cubes with small generators (Shanks):

- For the principal ideal class, the reduced ideal closest to

  $$\frac{R}{3} + \frac{\log(D')}{4}$$

- For any non-principal ideal class $C$, the three reduced ideals closest to

  $$d, \quad \frac{R}{3} + d, \quad \frac{2R}{3} + d$$

  where $0 < d < R/3$ and $z$ can be explicitly computed

  $(z$ depends on the representative of $C$)
Suitable Reduced Ideals, $D' < 0$

Principal infrastructure
Non-principal infrastructures
Shanks’ strategy for finding \( \lambda \) (or \( \overline{\lambda} \)):

- Search the infrastructures of \([a]\) and of \([\overline{a}]\) simultaneously to find \( \lambda \) or \( \overline{\lambda} \)
- The two infrastructures are mirror images of each other
In his 1990 PhD dissertation, Fung

- translated CUFFQI from Shanksian into a form suitable for computation
- implemented CUFFQI in Fortran on an Amdahl 5870 mainframe computer
- produced a number of examples, including the

\[
\frac{3^6 - 1}{2} = 364
\]

- cubic fields of the 19-digit discriminant

\[
D = -3161659186633662283
\]

in under 3 CPU minutes
CUFFFQI — Function Fields


Dictionary:

- $\mathbb{Q} \rightarrow \mathbb{F}_q(t)$, $q$ a prime power, $\gcd(q, 6) = 1$
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- \( \mathbb{Q} \rightarrow \mathbb{F}_q(t), q \) a prime power, \( \gcd(q, 6) = 1 \)
- \( \mathbb{Z} \rightarrow \mathbb{F}_q[x] \)
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- \( D \rightarrow D(t) \in \mathbb{F}_q[t] \) square-free
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- \( K = \mathbb{F}_q(t, x), \ [K : \mathbb{F}_q(t)] = 3 \)
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- $\mathbb{R} \rightarrow \mathbb{F}_q((x^{-1}))$
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- $\mathbb{R} \rightarrow \mathbb{F}_q((x^{-1}))$
- $\mathbb{C} \rightarrow \mathbb{F}_{q^2}((x^{-1}))$ or $\mathbb{F}_q((x^{-1/2}))$
Problems

- Infinite place of $\mathbb{F}_q(t)$ is archimedean — can decompose in any way
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- Hasse count is wrong
- There are three types of quadratic fields
Quadratic Function Fields

Let $D(t) \in \mathbb{F}_q[t]$ be squarefree

Let $\text{sgn}(D) \in \mathbb{F}_q^*$ denote the leading coefficient of $D(t)$. 
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\( \mathbb{F}_q(t, \sqrt{D}) \) is

- imaginary if \( \deg(D) \) is odd
  - infinite place of \( \mathbb{F}_q(t) \) ramifies
Quadratic Function Fields

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$\mathbb{F}_q(t, \sqrt{D})$ is

- **imaginary** if $\text{deg}(D)$ is odd
  - infinite place of $\mathbb{F}_q(t)$ ramifies
- **real** if $\text{deg}(D)$ is even and $\text{sgn}(D)$ is a square in $\mathbb{F}_q$
  - infinite place of $\mathbb{F}_q(t)$ splits
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  - infinite place of $\mathbb{F}_q(t)$ splits

- **unusual** if $\deg(D)$ is even and $\text{sgn}(D)$ is a non-square in $\mathbb{F}_q$
  - infinite place of $\mathbb{F}_q(t)$ is inert – no number field analogue!
Let $\mathbb{K}$ be a cubic extension of $\mathbb{F}_q(t)$ of square-free discriminant $D \in \mathbb{F}_q[t]$. 

deg($D$) odd: $\infty = pq^2$ in $\mathbb{K}$

deg($D$) even: $q \equiv 1 \pmod{3}$: $\text{sgn}(D) = 2$: $\infty = pqr$ or $p^3$ in $\mathbb{K}$ $\text{sgn}(D) \neq 2$: $\infty = pq$ or $p$ in $\mathbb{K}$

$q \equiv -1 \pmod{3}$: $\text{sgn}(D) = 2$: $\infty = pqr$ or $p$ in $\mathbb{K}$ $\text{sgn}(D) \neq 2$: $\infty = pq$ or $p^3$ in $\mathbb{K}$

Hasse count does not include the red cases.
Decomposition at Infinity in $K$

Let $K$ be a cubic extension of $\mathbb{F}_q(t)$ of square-free discriminant $D \in \mathbb{F}_q[t]$

Let $\infty$ denote the place at infinity in $\mathbb{F}_q(t)$. 

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$\deg(D)$ odd: $\infty = pq^2$ in $\mathbb{K}$

$\deg(D)$ even:

$q \equiv 1 \pmod{3}$:

$\sgn(D) = \Box$: $\infty = pq^2t$ or $p^3$ or $p$ in $\mathbb{K}$

$\sgn(D) \neq \Box$: $\infty = pq$ in $\mathbb{K}$

$q \equiv -1 \pmod{3}$:

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Hasse count does not include the red cases.

Renate Scheidler (Calgary)
Let $\mathbb{K}$ be a cubic extension of $\mathbb{F}_q(t)$ of square-free discriminant $D \in \mathbb{F}_q[t]$. Let $\infty$ denote the place at infinity in $\mathbb{F}_q(t)$.

$\text{deg}(D)$ odd: $\infty = pq^2$ in $\mathbb{K}$

$\text{deg}(D)$ even:

$q \equiv 1 \pmod{3}$:
- $\text{sgn}(D) = \Box$: $\infty = pq r$ or $p^3$ or $p$ in $\mathbb{K}$
- $\text{sgn}(D) \neq \Box$: $\infty = pq$ in $\mathbb{K}$

$q \equiv -1 \pmod{3}$:
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- $\text{sgn}(D) \neq \Box$: $\infty = pq$ or $p^3$ in $\mathbb{K}$

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As before, triples of conjugate cubic function fields are mapped onto pairs of 3-torsion ideal classes in $\mathbb{F}_q[t, \sqrt{D}]$. 
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For $\mathbb{F}_q(t, \sqrt{-3D})$ imaginary or unusual:
- bijection onto non-principal ideal classes
- nothing maps to the principal class

For $\mathbb{F}_q(t, \sqrt{-3D})$ real:
- 3-to-1 onto non-principal ideal classes
- 1-to-1 onto to the principal class
Some Counting

Put
\[ r = \text{3-rank}(\text{Cl}(\mathbb{Q}(\sqrt{D}))) \]
\[ s = \text{3-rank}(\text{Cl}(\mathbb{Q}(\sqrt{-3D}))) \]

Same field unless \( \text{deg}(D) \) even and \( q \equiv -1 \pmod{3} \)
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Number of cubic fields of discriminant \( D \) with at least two infinite places:
\[ \frac{3^r - 1}{2} \]
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Number of cubic fields produced by the Berwick map:

For \( \mathbb{F}_q(t, \sqrt{-3D}) \) imaginary or unusual:

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For \( \mathbb{F}_q(t, \sqrt{-3D}) \) real:

\[ \frac{3^{s+1} - 1}{2} \]
Some Counting

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\[ r = 3\text{-rank}(\text{Cl}(\mathbb{Q}(\sqrt{D}))) \]
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Connection between \( r \) and \( s \) (Lee 2007):

\[ |r - s| \leq 1 \]

If \( r \neq s \), then the unusual quadratic field has the bigger 3-rank
More Counting

If $\mathbb{F}_q(t, \sqrt{D}) = \mathbb{F}_q(t, \sqrt{-3D})$ (imaginary or real), then $r = s$  😊
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Case \( \mathbb{F}_q(t, \sqrt{-3D}) \) unusual, \( \mathbb{F}_q(t, \sqrt{D}) \) real:

\[
\begin{align*}
    r = s: & \quad \frac{3^s - 1}{2} = \frac{3^r - 1}{2} \\
    r = s - 1: & \quad \frac{3^s - 1}{2} = \frac{3^r - 1}{2} + 3^r
\end{align*}
\]
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So what are these extra $3^r$ cubic fields?
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- $r = s + 1$: $\frac{3^{s+1} - 1}{2} = \frac{3^r - 1}{2}$

So what are these extra $3^r$ cubic fields?

Answer: they are the fields with one infinite place that are missing from the Hasse count. In the ☹ cases, there are no such fields.
Reduced Ideals

The **genus** of $\mathbb{F}_q(t, \sqrt{-3D})$ is $\left\lfloor \frac{\deg(D) - 1}{2} \right\rfloor$. 

[Image of slide]
Reduced Ideals

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An ideal $\alpha$ in $\mathbb{F}_q(t, \sqrt{-3D})$ is reduced if $\deg(N(\alpha)) \leq g$. 
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An ideal $a$ in $\mathbb{F}_q(t, \sqrt{-3D})$ is reduced if $\deg(N(a)) \leq g$

Equivalent: $|N(a)| < \sqrt{|D|}$ where $|\cdot| = q^{\deg(\cdot)}$
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- a unique reduced ideal when $\mathbb{F}_q(t, \sqrt{-3D})$ is imaginary
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- either a unique reduced ideal or \( q + 1 \) “almost” reduced ideals (degree \( g + 1 \)) when \( \mathbb{F}_q(t, \sqrt{-3D}) \) is unusual (Artin 1924)
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(Almost) reduced ideals produce $\lambda$ with small norm: $|N(\lambda)| \leq |D|^{3/2}$
Suppose $\mathbb{F}_q(t, \sqrt{-3D})$ is imaginary or unusual.
Suppose $\mathbb{F}_q(t, \sqrt{-3D})$ is imaginary or unusual

Write $\lambda = A + B\sqrt{-3D}$ ($A, B \in \mathbb{F}_q[t]$). Then

$$N(\lambda) = A^2 + 3B^2D$$
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$$|N(\lambda)| \leq |D|^{3/2} \text{ implies }$$

$$|\text{Tr}(\lambda)| = |A| \leq |N(\lambda)|^{1/2} \leq |D|^{3/4}$$
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Yields again a small trace.
Suppose $\mathbb{F}_q(t, \sqrt{-3D})$ is real
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- Principal class: take reduced ideal closest to $\left\lceil R/3 + g/2 \right\rceil$
- Non-principal classes: take ideals closest to $d, R/3 + d, 2R/3 + d$ where $-g/2 \leq d < R/3 - g/2$ and $d$ can be explicitly computed using integer arithmetic only!
Example — Different 3-Rank

\[ q = 11, \quad D(x) = 7x^{10} + x^{7} + 3x^{6} + 2x^{5} + 7x^{4} + 8x^{3} + 4x^{2} + 2x \]
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\[ q = 11, \quad D(x) = 7x^{10} + x^7 + 3x^6 + 2x^5 + 7x^4 + 8x^3 + 4x^2 + 2x \]

\[ r = 3, \quad s = 2 \quad \Rightarrow \quad (3^3 - 1)/2 = 13 \text{ fields, all with } \infty = pq \text{ in } \mathbb{K}. \]
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\[ f(x) = x^3 - S(t)x + T(t) \text{ with} \]

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<td>1</td>
<td>5t^3 + 10t + 4</td>
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<tr>
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<tr>
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Example — Same 3-Rank

\[ q = 11, \quad D(x) = 2x^8 + x^6 + 5x^4 + 6x^2 + 7 \]
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\[ r = s = 2 \Rightarrow \begin{cases} 
(3^2 - 1)/2 = 4 & \text{fields with } \infty = pq \text{ in } \mathbb{K} \\
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BIG Example — Same 3-Rank

$q = 125, \quad D = 2x^{12} + 3x^9 + x^3 + 1$
BIG Example — Same 3-Rank

\[ q = 125, \quad D = 2x^{12} + 3x^9 + x^3 + 1 \]

\[ r = s = 5 \quad \Rightarrow \quad \left\{ \begin{array}{l}
(3^5 - 1)/2 = 121 \quad \text{fields with } \infty = pq \text{ in } K \\
3^5 = 243 \quad \text{fields with } \infty = p^3 \text{ in } K
\end{array} \right. \]
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364 fields
Concluding Remarks

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- CUFFQI can be extended to non-fundamental discriminants via basic class field theory and Kummer theory.

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Concluding Remarks

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- Ideas can be extended to higher degree fields with quadratic resolvent fields
Thank You — Questions?